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DYNAMIC AXISYMMETRIC BUCKLING OF SHALLOW CONES SUBJECTED TO IMPULSIVE LOADS

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SUMMARY

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A theoretical investigation is made of the axisymmetric snap-through buckling of a shallow cone subjected to an idealized impulse applied uniformly over the surface of the shell. The work is based on a study of the strain energy of the shell at a time when the displacement of the shell is a maximum (i.e., the velocity is zero). Under certain conditions this equilibrium position becomes unstable and the shell can snap-through (or buckle). Nonlinear strain-displacement equations are used and solutions are obtained for clamped and simply supported boundaries at the edge of the shell. Results for the cone are compared with similar results for a shallow spherical cap having the same rise as the cone. This comparison indicates that the sphere can resist a larger impulse than the cone before buckling.

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INTRODUCTION

The buckling of structures subjected to dynamic loads is of interest in many engineering problems. In the aerospace field, for example, the loads on missiles at launching and staging are cases where the principal forces are dynamic and where buckling can be the mode of failure. Unfortunately dynamic buckling is a much more difficult problem than is static buckling. Not only is there the added time parameter but also it is not yet clear what is meant by dynamic buckling and what constitutes a reasonable failure criteria. As a result of this situation numerous techniques have been used in an attempt to treat the problem.

A recent and useful method for studying dynamic buckling problems is one suggested by Humphreys and Bodner in references 1 and 2 and described in detail below. The method is limited to the snap-through buckling problem of a shell subjected to an initial idealized impulse; however, it can be very useful for this class of problems.

It is the purpose of this paper to apply the method of Humphreys and Bodner to the axisymmetric snap-through buckling of a shallow cone subjected to an idealized impulse applied uniformly over the surface of the shell. A solution is obtained for laterally restrained boundaries for both the simply supported and fixed cases. Solutions are also obtained for a rather unusual set of boundary conditions called "stress free", where both the horizontal and circumferential stresses are assumed to vanish. With these latter boundary conditions the analysis is simplified and results for the cone can be compared with similar results given in reference 1 for the shallow sphere.

SYMBOLS

b	external radius ~~half width~~ of the shell
r	horizontal coordinate
t	thickness of shell
u	horizontal displacement of shell
v_o	initial velocity of shell
w	maximum vertical displacement of shell
z	vertical coordinate
D	flexural stiffness of the shell, $\frac{Et^3}{12(1 - \mu^2)}$
E	Young's modulus
H	center rise of the shell
I	initial impulse parameter, $\frac{\bar{I}^2 b^4}{\rho t^2 E}$
I'	modified impulse parameter, $\frac{\bar{I}^2 b^4 3(1 - \mu^2)}{E\rho t^4 H^2}$, $\frac{I}{4\lambda^2}$
$\bar{I}$	initial impulse per unit area
K_1, K_2	radial ~~horizontal~~ and circumferential changes in curvatures
T_o	initial kinetic energy of shell
V	nondimensional strain energy, $\frac{\bar{V}}{2\pi H^2 tE}$
$\bar{V}$	strain energy of the shell
α	constant, see eq. (15)
ϵ_1, ϵ_2	radial ~~horizontal~~ and circumferential extensional strains
λ	shell parameter, $\frac{H}{t}\sqrt{12(1 - \mu^2)}$
λ'	modified shell parameter, $\sqrt{2\lambda}$, $\sqrt{\frac{H}{t}}\left[48(1 - \mu^2)\right]^{1/4}$
μ	Poisson's ratio
σ_1, σ_2	horizontal and circumferential stresses
ρ	mass density of the shell

METHOD OF APPROACH

In treating the problem of the snap–through buckling of a shell subjected to an impulse, it is assumed that the load is distributed uniformly over the structure. Furthermore, since the load is an impulse it can be considered to be removed before the shell has begun to move. This impulse therefore transmits a uniform initial velocity to the structure; that is, it represents an input kinetic energy. At some later time, t_1 (not determined), it is assumed that the displacement of all points on the shell has reached a maximum and the velocity of these points is zero, hence the shell exhibits the character of a single degree of freedom. The snap–through ~~character~~ behavior of the structure can be determined from a study of the equilibrium state at time t_1. At this time the internal strain energy of the shell must be equal to the initial kinetic energy because of conservation of energy. From a study of the strain energy at time t_1, one can then determine the initial kinetic energy required to cause the motion of the shell to exhibit a large jump. This jump is then defined as the point of snap–through or "buckling".

A qualitative plot of the strain energy versus deflection of the structure is shown in figure 1. A requirement for "buckling" as defined herein is that the strain energy of the shell must decrease with increase in maximum deflection. ~~(indicating an increase in kinetic energy since the total energy is constant)~~. Consequently, when snap–through occurs the strain energy at maximum displacement must be a relative maximum with respect to the maximum displacement. ~~Thus at "buckling" the rate of change of the strain energy with respect to the maximum displacement must vanish.~~

An alternate explanation of the snap–through criteria is to note that the initial kinetic energy is equal to the strain energy at the maximum displacement w_m. Hence, the maximum displacement can be regarded as a function of the initial input velocity. When the shell snaps through there is a sudden increase in maximum displacement with respect to a change in initial velocity. The condition for snap-through is therefore given by

$$\frac{dw_{\cancel{m}}}{dv_o} = \infty$$

or

$$\frac{dv_o}{dw_{\cancel{m}}} = 0$$

GOVERNING EQUATIONS

The problem to be considered is that of a shallow cone subjected to axisymmetric deformations. The cone has a height H, an external radius b, and thickness t. The radius to a point on the middle surface is r (see fig. 2). The shallow cone considered here is one where the rise angle of the cone and its sine can be replaced by H/b.

The strain energy $\overline{V}$ of the cone at some time when the displacement is a maximum and the velocity is zero can be expressed as the sum of the contribution due to membrane effects $\overline{V}_m$ and bending effects $\overline{V}_b$. Thus

$$\overline{V} = \overline{V}_m + \overline{V}_b \tag{1a}$$

where

$$\overline{V}_m = \frac{\pi E t}{(1 - \mu^2)} \int_0^b \left(\epsilon_1{}^2 + \epsilon_2{}^2 + 2\mu\epsilon_1\epsilon_2\right) r dr \tag{1b}$$

$$\overline{V}_b = \pi D \int_0^b \left(K_1{}^2 + K_2{}^2 + 2\mu K_1 K_2\right) r dr \tag{1c}$$

Here ϵ_1, ϵ_2 and K_1, K_2 are, respectively, the extensional strains and curvatures in the radial and circumferential direction. Young's modulus is denoted by E, Poisson's ratio by μ, and the shell stiffness by $D = \frac{Et^3}{12(1-\mu^2)}$.

The strain energies can be nondimensionalized as

$$V = V_m + V_b \tag{2a}$$

where

$$V = \frac{\overline{V}}{2\pi H^2 tE}, \text{ etc.} \tag{2b}$$

resulting in

$$V_m = \frac{1}{2(1-\mu^2)H^2}\int_0^b \left(\epsilon_1{}^2 + \epsilon_2{}^2 + 2\mu\epsilon_1\epsilon_2\right)rdr \tag{3a}$$

and

$$V_b = \frac{1}{2\lambda^2}\int_0^b \left(K_1{}^2 + K_2{}^2 + 2\mu K_1 K_2\right)rdr \tag{3b}$$

with

$$\lambda^2 = \frac{12H^2(1-\mu^2)}{t^2} \tag{4}$$

The usual stress–strain relations taken for the ~~horizontal~~ radial and ~~vertical~~ circumferential in plane stresses are, respectively

$$\sigma_1 = \frac{E}{(1-\mu^2)}\left(\epsilon_1 + \mu\epsilon_2\right)$$

$$\sigma_2 = \frac{E}{(1-\mu^2)}\left(\epsilon_2 + \mu\epsilon_1\right) \tag{5}$$

The nonlinear strains for axisymmetric deformations of the shallow cone are taken as

$$\epsilon_1 = u_r + \frac{H}{b} w_r + \frac{1}{2} w_r^2 \tag{6a}$$

$$\epsilon_2 = \frac{u}{r} \tag{6b}$$

$$K_1 = w_{rr} \tag{6c}$$

$$K_2 = \frac{w_r}{r} \tag{6d}$$

where u and w are the deflections in the horizontal (r) and vertical (z) directions, respectively, and the subscript r denotes differentiation with respect to r.

The compatibility relationship between the strains becomes, therefore

$$(\epsilon_2 r)_r - \epsilon_1 = - \frac{1}{2} w_r^2 - \frac{H}{b} w_r \tag{7}$$

If the velocity in the r direction $\frac{\partial u}{\partial t}$ is neglected, the equation of radial ~~horizontal~~ equilibrium can be obtained from the first variation of equation (3a) to give

$$\frac{d}{dr}(r\sigma_1) - \sigma_2 = 0 \tag{8}$$

Equations (5) and (6) can be used to write equation (8) giving

$$u_{rr} + \frac{1}{r} u_r - \frac{u}{r^2} = -w_{rr} w_r - \frac{(1-\mu)}{2} \frac{w_r^2}{r} - (1-\mu) \frac{H}{b} \frac{w_r}{r} - \frac{H}{b} w_{rr} \tag{9}$$

Taking account of equation (8) it can be shown that the following useful identity exists between the stresses and strains.

$$\frac{d}{dr}(r\epsilon_2) - \epsilon_1 = \frac{1}{E}\left[\frac{d}{dr}(r\sigma_2) - \sigma_1\right] \tag{10}$$

IMPULSE PARAMETER

It is assumed that the shell undergoes an initial uniformly distributed velocity due to an impulse per unit area $\bar{I}$. The initial kinetic energy T_o can therefore be related to the impulse by

$$T_o = \frac{\pi b^2 \bar{I}^2}{2\rho t} \tag{11}$$

where ρ is the mass density of the shell.

At some later time, t_1, when the velocity is zero and the displacement has reached a maximum the strain energy $\bar{V}$ at t_1 is equal to the initial kinetic energy. Hence

$$\bar{V} = \frac{\pi b^2 \bar{I}^2}{2\rho t} \tag{12}$$

Nondimensionalizing according to equations (2), there results

$$4\, b^2/H^2\, V = 4\, b^2/H^2 (V_m + V_b)$$

$$I = \cancel{4\lambda^2 V = 4\lambda^2 (V_m + V_b)} \tag{13}$$

where

$$I = \cancel{\frac{\bar{I}^2\lambda^4}{\rho t^2 E}} \quad \frac{\bar{I}^2 b^4}{\rho t^2 E} \tag{13a}$$

a nondimensional impulse parameter and where V_m and V_b are given by equations (3a) and (3b).

RIGID BOUNDARY

Consider first the problem where the edge of the shell at $r = b$ is restrained from horizontal movement (rigid), i.e.,

$$w/_{r=b} = u/_{r=b} = 0 \tag{14}$$

The method of approach is to assume a vertical deflection $w(r)$ and to determine the impulse I given by equation (13) in terms of this deflection. The impulse has been shown to be related to the strain energy which is a function of both the lateral displacement u as well as w. The lateral displacement u can be determined for a known w by integrating the equation of horizontal equilibrium, equation (9), together with the boundary condition, equation (14), and the fact that $u = 0$ at $r = 0$. Consequently, the initially applied impulse I is expressible in terms of only w, the maximum displacement. The impulse required for snap-through can then be determined by differentiating I with respect to w.

An approximate solution for the maximum deflected shape can be taken as

$$w = w_o H\left(1 - \frac{r^2}{b^2}\right)\left(1 - \alpha\frac{r^2}{b^2}\right) \tag{15}$$

where w_o and α are constants. The boundary conditions at $r = b$ may be satisfied by a proper choice of α, i.e.

Clamped Edge: $w_r = 0, \qquad \alpha = 1$

Simply Supported: $M_1 = -D(K_1 + \mu K_2) = 0, \qquad \alpha = \dfrac{1 + \mu}{5 + \mu}$

The shape given by equation (15) corresponds to the shape taken by a circular plate subjected to a uniform lateral load and supported at the boundary.

By substituting equation (15) into equation (13) and taking account of equation (9) and the appropriate boundary conditions on u there results after some effort and with $\mu = 0.3$

1. For the Rigid Clamped Edge ($w_r/_{r=b} = u/_{r=b} = 0, \quad \alpha = 1$)

$$I = 4\lambda^4\left[0.1151w_o^{\,4} - 0.3633w_o^{\,3} + w_o^{\,2}\left(0.2876 + \frac{7.1333}{\lambda^2}\right)\right] \quad (16)$$

2. For the Rigid Simply Supported Edge ($M_1/_{r=b} = u/_{r=b} = 0, \ \alpha = 0.24528$)

$$I = 4\lambda^4\left[0.1527w_o^{\,4} - 0.4599w_o^{\,3} + w_o^{\,2}\left(0.3485 + \frac{2.0767}{\lambda^2}\right)\right] \quad (17)$$

The condition for snap-through is

$$\frac{dI}{dw_o} = 0 \quad (18)$$

Carrying out the differentiation and taking the smaller root of the resulting quadratic in w_o yields

1. Clamped Edge

$$w_o = 1.18364 - \sqrt{0.15165 - \frac{30.98755}{\lambda^2}} \quad (19)$$

2. Simply Supported Edge

$$w_o = 1.12942 - \sqrt{0.13446 - \frac{6.79993}{\lambda^2}} \quad (20)$$

It should be noted that at certain values of λ the amplitude w_o becomes imaginary. At such values of λ there is no real w_o which correspond to the snap-through condition. The physical interpretation of imaginary values of w_o would appear to be that the shell does not exhibit a snapping phenomenon and the motion is a smooth oscillatory motion. The minimum values of λ corresponding to real w_o are

1. Clamped Edge $\qquad\qquad \lambda_{min} = 14.295$

   i.e., for $\mu = 0.3$, $\qquad\qquad H > 4.34t$

2. Simply Supported $\qquad\qquad \lambda_{min} = 7.111$

   i.e., for $\mu = 0.3$, $\qquad\qquad H > 2.88t$

Thus, for example, in the case of a cone with a rigidly fixed boundary snap-through buckling does not occur unless the rise of the cone H exceeds 4.34t.

A plot of equations (16) and (17) taking account of equations (19) and (20) is given in figure 3. It should be noted that a slight change in parameters has been used in the plot in order to make some comparison with the previously published results for a sphere. The modified parameters are λ' and I' where

$$\lambda' = \sqrt{2\lambda} = \sqrt{\frac{H}{t}}\left[48(1-\mu^2)\right]^{1/4} \tag{21}$$

$$I' = \frac{I}{4\lambda^2} = \frac{\bar{I}^2 b^4 12(1-\mu^2)}{E\rho t^4 H^2} \tag{22}$$

Figure 4 gives a plot of λ' versus the maximum critical deflection to thickness ratio which exists at the time of buckling. A study of the magnitude of this critical deflection shows that its value is of the order of the rise of the cone. In fact for $\lambda' = 7.75$ and 11.0, $\frac{w_o}{H}$ for the clamped case is 0.840 and 0.805, respectively.

STRESS-FREE BOUNDARY

The effort required to obtain the impulse in terms of an assumed displacement can be reduced considerably if a rather unusual set of boundary conditions are taken at the edge $r = b$. These conditions are denoted as "stress free" and correspond to the case where

$$w/_b = \sigma_1/_b = \sigma_2/_b = 0$$

With these conditions it can be shown that it [illegible]ecessary to solve for u in terms of w.

While these boundary conditions may not appear realistic, the way in which they simplify the calculations makes [illegible]rthy of consideration. They were also treated for similar reasons [illegible]ference 1 for the spherical cap and, consequently, are of interest here in order to compare the results for the cone with those of the sphere.

The membrane strain energy equation (3a) can also be written in terms of the stresses as

$$V_m = \frac{1}{2H^2E}\int_o^b \left(\sigma_1^2 + \sigma_2^2 - \mu\sigma_1\sigma_2\right) r dr \tag{24}$$

The product term $\sigma_1\sigma_2$ in equation (24) makes no contribution to the energy for the stress-free boundary at b as is shown below. Using equation (8) there results

$$r\sigma_2\sigma_1 = r\sigma_1(r\sigma_1)_r$$

Integrating over the limits of r yields

$$\int_o^r r\sigma_2\sigma_1 dr = \int_o^b r\sigma_1(r\sigma_1)_r \, dr = \left(\frac{r\sigma_1}{2}\right)^2\Bigg]_o^b = 0 \tag{25}$$

Consequently, equation (24) can be written as

$$V_m = \frac{1}{2H^2E}\int_o^b \left(\sigma_1 + \sigma_2\right)^2 r \, dr \tag{26}$$

Also from the compatibility equation (7) together with equation (10) it can be shown that

$$\frac{1}{E}\left(\sigma_1 + \sigma_2\right)_r = -\frac{1}{2}\frac{1}{r}w_r^2 - \frac{H}{b}\frac{w_r}{r} \tag{27}$$

Integrating equation (27) from r to b and considering the stress-free boundary conditions (eq. (23)) yields

$$\frac{1}{E}\left(\sigma_1 + \sigma_2\right) = \int_r^b \left(\frac{1}{r}w_r^2 + \frac{H}{b}\frac{w_r}{r}\right)dr \tag{28}$$

From equation (28) the membrane energy equation (26) can now be expressed in terms of only the vertical deflection w as follows

$$V_m = \frac{1}{2H^2}\int_o^b \left[\int_r^b \left(\frac{1}{x}w_x^2 + \frac{H}{b}\frac{w_x}{x}\right)dx\right]^2 r\, dr \tag{29}$$

where x is a dummy variable of integration.

Again assuming w to be in the form of equation (15) and taking account of equation (29) in equation (13) the nondimensional impulse parameter I becomes with $\mu = 0.3$.

1. Clamped Stress-Free Edge $w_r = \sigma_1 = \sigma_2 = 0$, $\alpha = 1$

$$I = 4\lambda^4\left[0.06349w_o^{\,4} - 0.19817w_o^{\,3} + w_o^{\,2}\left(0.15556 + \frac{5.333}{\lambda^2}\right)\right] \tag{30}$$

2. Simply Supported Stress-Free Edge ($M_1 = \sigma_1 = \sigma_2 = 0$, $\alpha = 0.24528$)

$$I = 4\lambda^4\left[0.0675w_o^{\,4} - 0.2078w_o^{\,3} + w_o^{\,2}\left(0.1619 + \frac{1.6309}{\lambda^2}\right)\right] \tag{31}$$

The corresponding critical deflections at snap-through buckling together with the limits on λ are

1. Clamped Stress-Free Edge

$$w_o = 1.17045 - \sqrt{0.14496 - \frac{42}{\lambda^2}} \qquad (32)$$

$$\lambda_{min} = 17.0216$$

$$H > 5.16t$$

2. Simply Supported Stress-Free Edge

$$w_o = 1.1544 - \sqrt{0.1334 - \frac{12.08078}{\lambda^2}} \qquad (33)$$

$$\lambda_{min} = 9.516$$

$$H > 2.88t$$

Plots of the modified critical impulse I' versus the modified shell parameter λ' defined by equations (21) and (22) are given in figure 3. Figure 4 shows a plot of the amplitude of the critical deflection versus λ'.

DISCUSSION OF RESULTS

It is interesting to note in figure 3 that for relatively large values of λ the strength of the simply supported shell is greater than the fixed case. On the other hand for small values of λ where the rise of the cone is quite small, the reverse effect occurs. These results would seem to indicate that for high rise cones (relatively speaking) where the membrane stresses are important, the more flexible the shell the stronger it is to resist dynamic effects. In the case of small rises where the cone is very nearly a plate the major strength comes from the boundary conditions and the bending rigidity of the shell. In this case the rigid boundaries are an aid to the dynamic strength of the shell.

The first effect has been noted previously by Humphreys and Bodner (ref. 1) for long cylindrical panels where the simply supported panels exhibited a greater resistance to snap-through than did clamped panels. The reversal in the relative strengths for the two boundary conditions which occurs for the very shallow cones, however, did not occur for the long cylindrical panels.

It is also of interest to compare the behavior of a shallow cone with that of a shallow sphere of comparable geometry. In order to make this comparison it was necessary to express the sphere parameter in terms of the rise of the shell using the approximate relations appropriate for a shallow sphere. This was done and figures 5 and 6 show the results for comparable boundary conditions. The spherical results were taken from figures 6 and 7 of reference 1. In general, figure 5 shows that for a conical or spherical shell of the same rise H, the stronger structure to resist dynamic buckling is the sphere. Figure 6 indicates that at buckling the sphere has also undergone a larger deflection than the comparable cone.

A final comment should be made with respect to the accuracy of the present results. The two major sources of error lie in the choice of a single degree of freedom for the deflected shape and the requirement that the cone deform axisymmetrically. Both of these effects become more important with increasing shell rise and the latter effect would seem to be the more important of the two. In view of these effects the results given herein are inadequate for λ' beyond about 10; however, they may be useful as qualitative data beyond that range. It should also be noted that it was shown in reference 3 that the results of a single degree of freedom approximation for a shallow sphere are unconservative for large λ'. It seems reasonable to expect the same qualitative behavior to hold for the shallow cone.

ACKNOWLEDGEMENT

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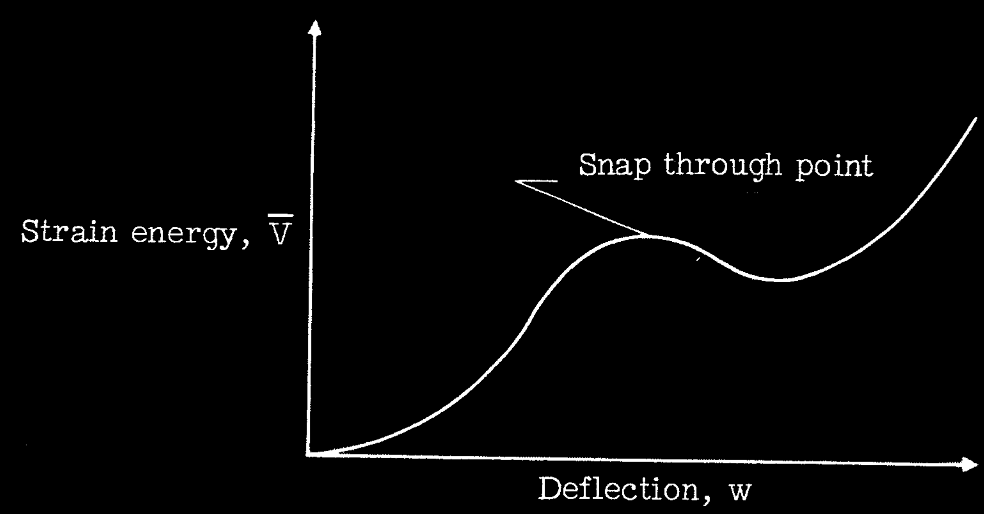

Figure 1.- Plot of strain energy versus maximum deflection.

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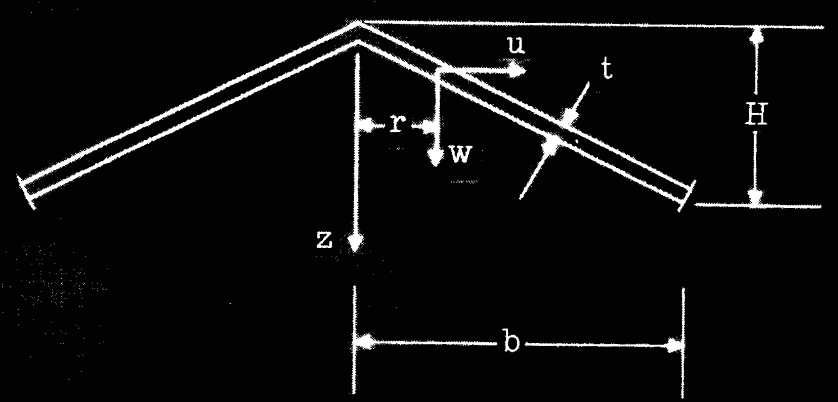

Figure 2. - Shell geometry for axisymmetric cone.

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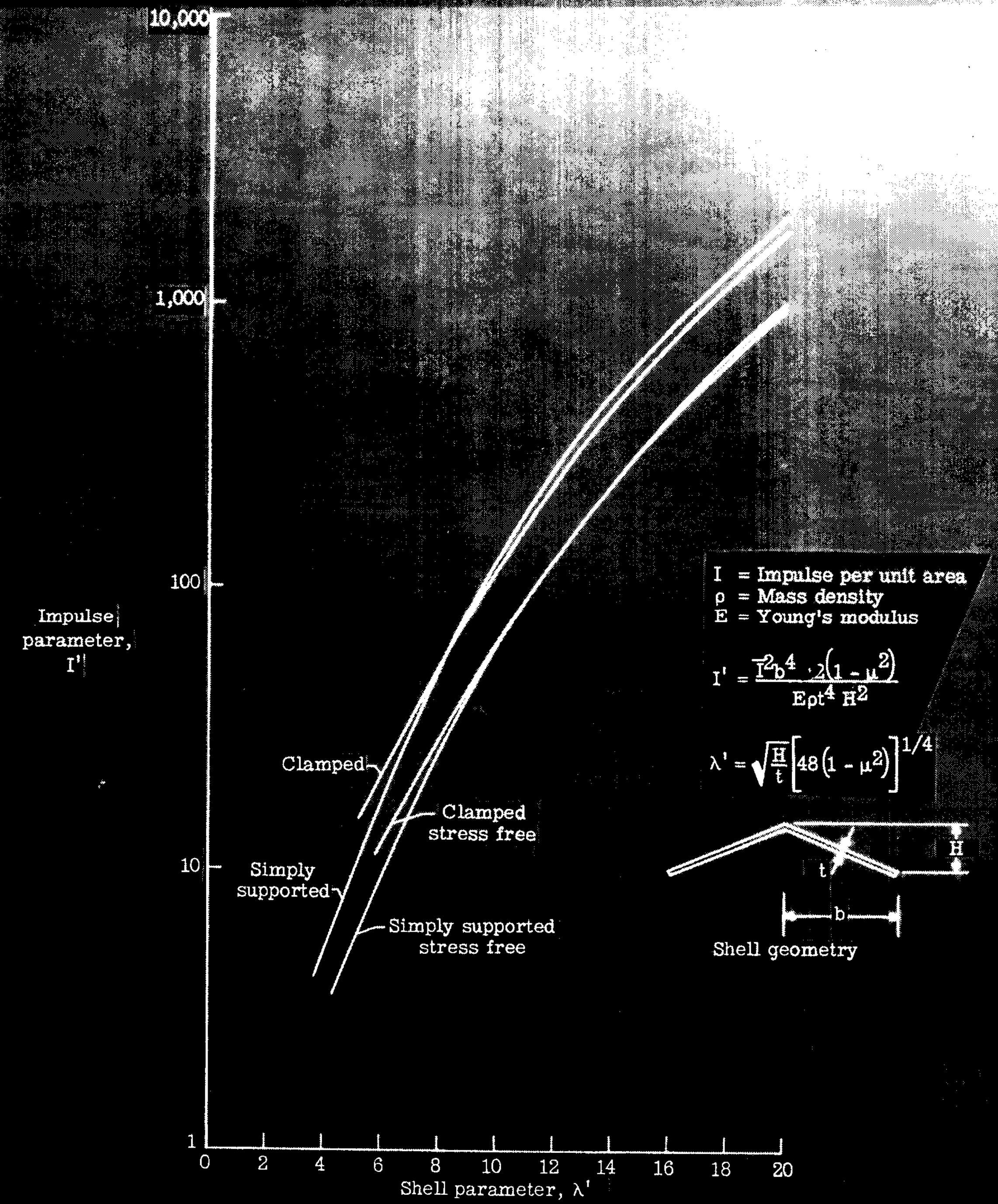

Figure 3. - Snap through impulse versus shell parameter for axisymmetric shallow cones.

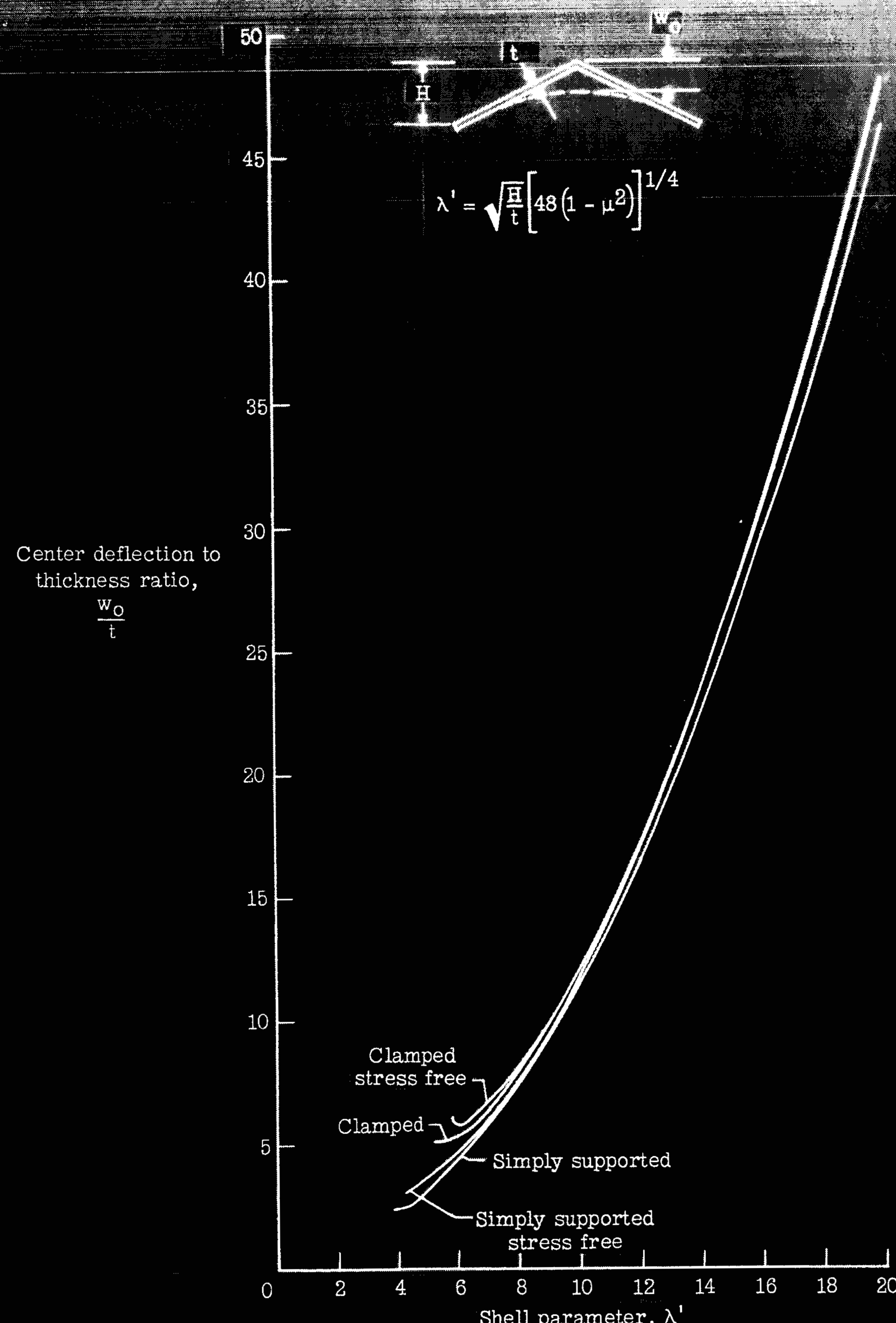

Figure 4. - Deflection at snap through for axisymmetric shallow cones subjected to impulsive loads.

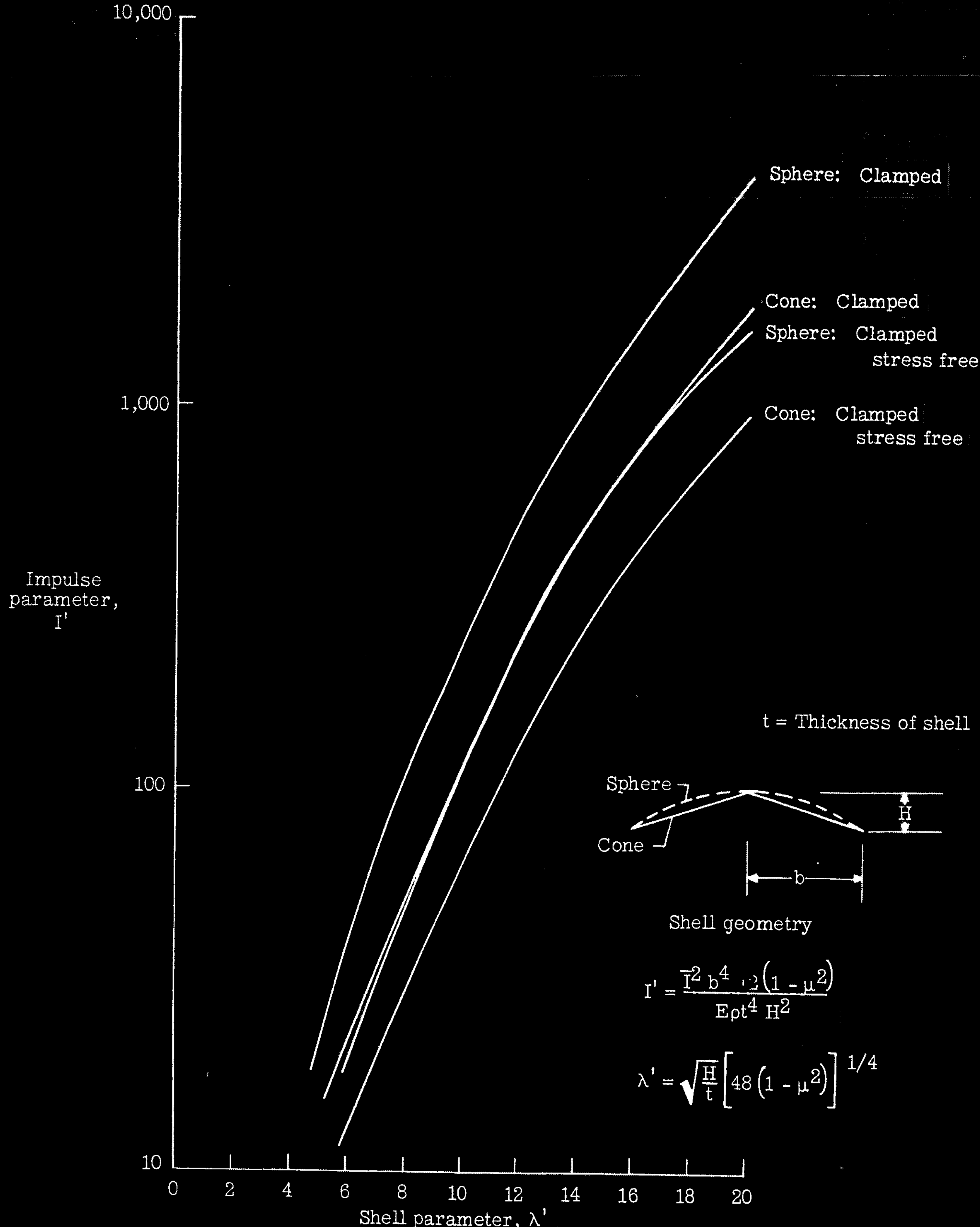

Figure 5. - Snap through impulse versus shell parameter for clamped axisymmetric shallow cones and spheres.

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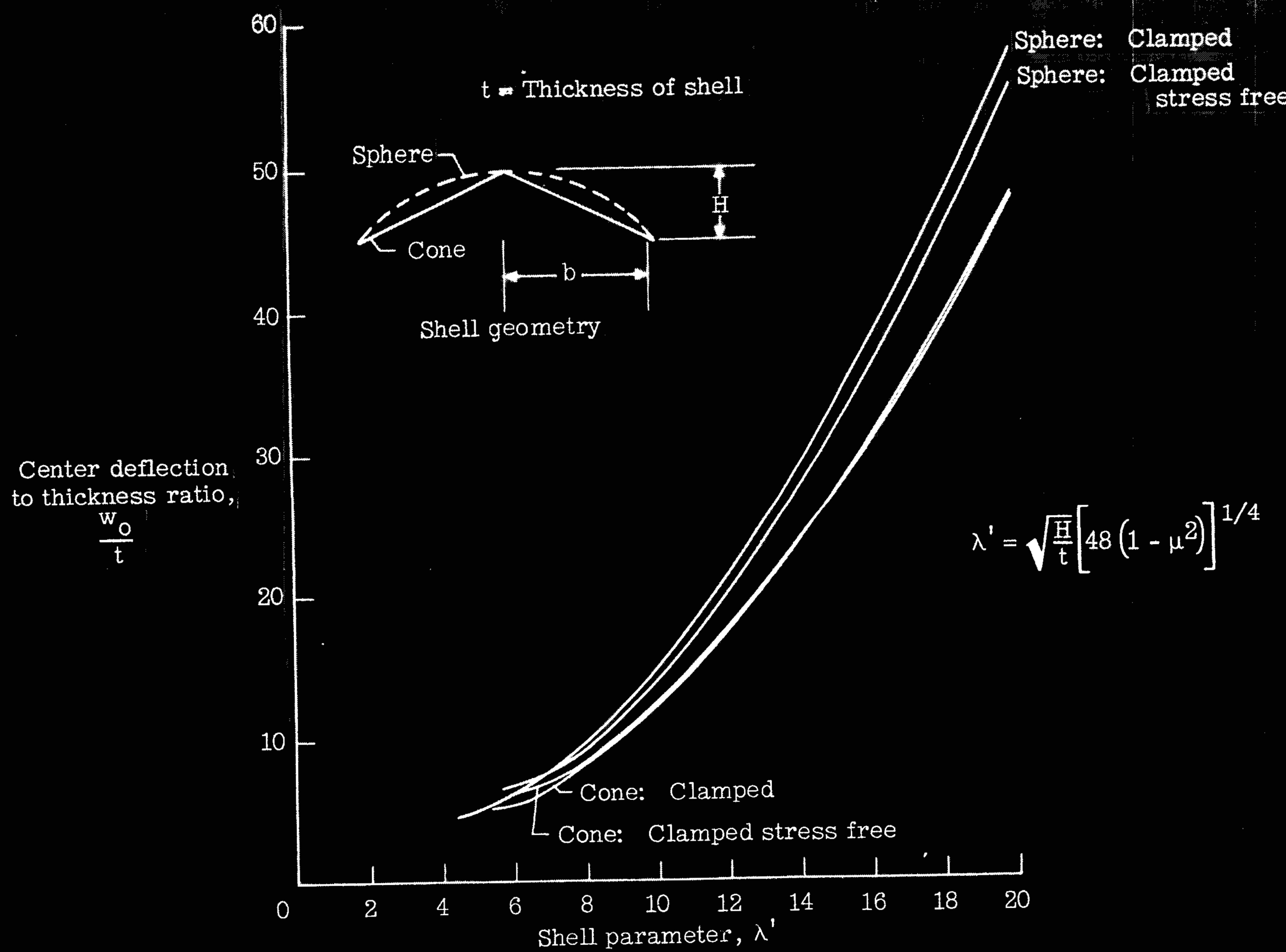

Figure 6.- Deflection at snap through for clamped axisymmetric shallow cones and spheres subjected to impulsive loads.